

# Generalized Chern-Simons Modified Gravity in First-Order Formalism

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(Dated: December 8, 2009)

We propose a generalization of Chern-Simons (CS) modified gravity in first-order formalism. CS modified gravity action has a term that comes from the chiral anomaly which is Pontryagin invariant. First-order CS modified gravity is a torsional theory and in a space-time with torsion the chiral anomaly includes a torsional topological term called Nieh-Yan invariant. We generalize the CS modified gravity by adding the Nieh-Yan term to the action and find the effective theory. We compare the generalized theory with the first-order CS modified gravity and comment on the similarities and differences.

PACS numbers: 04.50.Kd; 04.20.Fy; 04.60.Cf

## I. INTRODUCTION

The modifications of General Relativity (GR) are widely considered in the literature to find the true high energy limit of it. One way of modification is adding some topological terms to the Einstein-Hilbert Lagrangian. In four dimensions the topological invariants related to the curvature are Pontryagin and Euler classes. These are added to the action with associated topological parameters which are considered as coupling constants [1]. Since the Pontryagin class is an exact form and can be written in terms of the Chern-Simons (CS) form, in the case of constant coupling it can be written as a boundary term and the equations of motion do not change. However, the boundary terms change the canonical variables in the Hamiltonian formalism of the theory and can lead to different ways in canonical quantization of gravity [2]. Another topological modification is adding the Holst term to the action with constant coupling parameter called the Immirzi parameter [3, 4]. Although the Holst term is not a topological invariant, it can be considered as the 'half' of a torsional topological invariant which is mentioned below.

By adding more external fields to the theory one can reach another type of modification. One way of doing this is promoting the coupling constant of a topological term to a field. Adding the Pontryagin term to the action by multiplying a space-time dependent coupling scalar field leads to the CS modification of GR [5]. In CS modified gravity the field equations include an extra term which is a Cotton-like tensor that is formed by Ricci tensor and the dual of the Riemann tensor. So, the field equations are higher than second order in derivatives. But, this does not prevent the Schwarzschild, Robertson-Walker and gravitational wave space-times from being solutions of CS modified gravity [5]. However, the theory has a constraint resulting from the field equations that

are found from varying the external scalar field. This constraint requires the vanishing of the Pontryagin term. For a recent review of CS modified gravity, see [6]. Different aspects of it are also investigated in the literature [7, 8, 9, 10].

Recently, the first-order formalism of the CS modified gravity was considered by several people [11, 12]. The main motivation for this was that treating the curvature of the underlying manifold as a gauge curvature of the connection that can be considered as a gauge variable. In pure first-order theory the equations of motion leads to zero torsion and the theory reduces to Einstein field equations. However, in the presence of matter with spin, the theory leads to non-zero torsion. CS modified gravity is written in first-order formalism as adding the Pontryagin term with a coupling scalar field to the first-order action. Although the equations of motion found from varying the co-frame field leads to the same field equations as in pure theory, the equations of motion found from varying the connection leads to a non-zero torsion. So, in first-order formalism, CS modified gravity is a torsional theory and the torsion is written in terms of the curvature 2-forms and derivatives of the coupling scalar field.

In the presence of torsion, besides the Pontryagin and Euler classes there is a torsional topological invariant called Nieh-Yan class [13, 14]. So, by coupling with an external scalar field, one can use it for a modification of GR in a torsional geometry. Similar to the Pontryagin term, Nieh-Yan class can also be written as an exact form in terms of a CS-like term. This is called the translational CS form [15]. Since first-order formulation of CS modified gravity has torsion, in the sake of generality, one must also use the Nieh-Yan class in the topological coupling term. By doing this, we reach a torsional generalization of CS modified gravity. But, in this theory, field equations will differ from the pure CS modified gravity and torsion will have an additional term. Moreover, coupling the theory with Dirac fermions leads to an effective action that includes more interaction terms than before.

CS modified gravity is also used as a cancelation mechanism for Green-Schwarz anomaly since the Green-Schwarz mechanism requires the inclusion of a Pontrya-

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gin term to the action [16]. Existence of anomalies are related to the topological properties of the background. In torsion-free case the chiral anomaly is proportional to the Pontryagin class of the manifold. However, in the presence of torsion, the chiral anomaly has an additional term which is proportional to the Nieh-Yan class [17, 18, 19, 20]. This is another motivation for generalizing the CS modified gravity to include the Nieh-Yan term. On the other hand, the SO(4) Nieh-Yan class is included in the SO(5) Pontryagin class [17]. By thinking the embedding of SO(4) into SO(5) and treating gravity as an SO(5) gauge theory, the modification with an SO(5) Pontryagin term leads to an SO(4) Nieh-Yan term.

In this paper, we add the Nieh-Yan term to the Pontryagin term in the CS modified gravity. The field equations of generalized CS modified gravity are found. By coupling with fermions, the torsion and contorsion forms are obtained and the effective action is written. So, the new interaction terms are found and the physical interpretations are discussed.

## II. CS MODIFIED GRAVITY

In its original version, CS modified gravity is considered in second-order formalism [5, 7, 8]. The action is written as a sum of the Einstein-Hilbert term and the Pontryagin term with a coupling scalar field;

$$S = \kappa \int d^4x \sqrt{-g} (R + \frac{1}{4} \theta^* RR)$$

where  $\kappa = \frac{1}{16\pi G}$ ,  $g$  is the determinant of the metric,  $R$  is the curvature scalar,  $\theta$  is the coupling field and  $*RR$  is the Pontryagin density which is defined by

$$*RR = *R^{abcd} R_{bacd}$$

and the dual Riemann tensor is  $*R^{abcd} = \frac{1}{2} \epsilon^{cdef} R_{abef}$  with  $\epsilon^{abcd}$  is totally antisymmetric Levi-Civita tensor. The Pontryagin density can be written as a total derivative of the CS term. Hence, by taking integration by parts and neglecting the boundary terms, the second term of the action can be written as follows

$$S_{CS} = -\frac{\kappa}{2} \int d^4x \sqrt{-g} v_a K^a$$

where  $v_a = \nabla_a \theta = \partial_a \theta$  is the CS velocity and  $K^a$  is the CS term;

$$K^a = \epsilon^{abcd} \Gamma_{bf}^e (\frac{1}{2} R_{cde}^f - \frac{1}{3} \Gamma_{ce}^k \Gamma_{dk}^f)$$

which satisfies  $\nabla_a K^a = *RR/2$  with  $\nabla_a$  is the covariant derivative with respect to the Levi-Civita connection  $\Gamma_{ab}^c$  [5].

The field equations are given by

$$G_{ab} + C_{ab} = 0$$

where  $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$  is the Einstein tensor and  $C_{ab}$  is the Cotton-like tensor which is defined by

$$C_{ab} = -\frac{1}{2\sqrt{-g}} (v_a (\epsilon^{abcd} \nabla_c R^e_d + \epsilon^{aecd} \nabla_c R^b_d) + \nabla_a v_f (*R^{fabe} + *R^{feab})).$$

On the other hand, variation with respect to the coupling scalar field  $\theta$  yields a constraint equation which is called Pontryagin constraint;

$$*RR = 0.$$

This implies that the solutions of CS modified gravity must be of Petrov types *III*, *N* and *O* while the space-times of Petrov types *D*, *II* and *I* does not satisfy it [6]. So, the Schwarzschild, Robertson-Walker and gravitational wave space-times are solutions of the CS modified gravity.

## III. FIRST-ORDER FORMALISM

In first-order formalism of gravity, independent field variables are the orthonormal co-frame basis  $e^a$  and the connection 1-forms  $\omega^{ab}$ . Torsion and curvature 2-forms are defined from these quantities respectively as;

$$\begin{aligned} T^a &= de^a + \omega^a_b \wedge e^b \\ R^{ab} &= d\omega^{ab} + \omega^a_c \wedge \omega^{cb} \end{aligned}$$

where  $d$  is the exterior derivative. Torsion and curvature 2-forms can be used in the construction of topological invariants. In four dimensions, the topological invariants defined from curvature are Pontryagin and Euler classes. These are written as

$$P = \frac{1}{8\pi^2} \int R^{ab} \wedge R_{ab} \quad (1)$$

$$E = \frac{1}{16\pi^2} \int R^{ab} \wedge *R_{ab} \quad (2)$$

where  $*$  is the Hodge star operator on differential forms. CS modified gravity in first-order formalism is constructed from Einstein-Hilbert action and the Pontryagin term coupled with an external scalar field  $\theta$ ;

$$\begin{aligned} S &= S_{EH} + S_{CS} \\ &= \kappa \int R^{ab} \wedge *e_{ab} + \frac{1}{2} \int \theta R^{ab} \wedge R_{ab}. \end{aligned} \quad (3)$$

Here  $e_{ab} = e_a \wedge e_b$ . However, the Pontryagin term can be written as a boundary term, namely exterior derivative of the CS form;

$$R^{ab} \wedge R_{ab} = d(\omega^{ab} \wedge R_{ab} - \frac{1}{3} \omega^{ab} \wedge \omega_a^c \wedge \omega_{cb}).$$

So, the second term in the action can be written as

$$S_{CS} = -\frac{1}{2} \int d\theta \wedge (\omega^{ab} \wedge R_{ab} - \frac{1}{3} \omega^{ab} \wedge \omega_a^c \wedge \omega_{cb}). \quad (4)$$

By varying the action with respect to  $e^a$  and  $\omega^{ab}$ , the field equations in first-order formalism are found as

$$\begin{aligned} G_a &= R^{cb} \wedge *e_{cba} = 0 \\ \kappa D * e_{ab} + d\theta \wedge R_{ab} &= 0 \end{aligned}$$

where  $G_a$  is Einstein 3-forms and  $D$  is the covariant exterior derivative. The second equation requires that the torsion is not zero. Moreover, variation with respect to the coupling scalar field leads to a constraint that requires the Pontryagin class to be zero;

$$P = R^{ab} \wedge R_{ab} = 0.$$

Hence, in first-order formalism, CS modified gravity is a torsional theory and this has some implications in the case of interaction with fermions.

### A. Interaction with Fermions

In the presence of fermions, the coupled action of CS modified gravity is

$$S = S_{EH} + S_{CS} + S_D$$

where the Dirac action is written as

$$S_D = \frac{ik}{2} \int *e_a \wedge (\bar{\psi} \gamma^a D\psi - \overline{D\psi} \gamma^a \psi). \quad (5)$$

Here  $\psi$  is the fermion field,  $\gamma^a$  are gamma matrices and

$$D\psi = d\psi + \omega\psi$$

where  $\omega = (1/4)\omega^{ab}\gamma_a\gamma_b$ . In the first field equations, the only difference is the inclusion of the stress-energy forms of matter  $\tau^a$  which are variations of the Dirac action with respect to the coframe field  $e^a$ . The second field equations are changed as

$$\kappa D * e_{ab} + d\theta \wedge R_{ab} + \frac{k}{4} A^c e_{abc} = 0. \quad (6)$$

Here

$$A^c = \bar{\psi} \gamma_5 \gamma^c \psi$$

is the axial current. In this case, the torsion 2-forms are found as below [6]

$$T_a = \frac{1}{2\kappa} v^b \epsilon_{bapq} R^{pq} + \frac{k}{8\kappa} A^b \epsilon_{bapq} e^{pq} + O(v^2) \quad (7)$$

where  $i_X$  is the interior product with respect to the vector field  $X$  and by defining  $v^a = i_{X^a} d\theta$  we neglect the terms that are second order in  $v$ . On the other hand, contorsion 1-forms are defined as the difference between the connection 1-forms  $\omega^{ab}$  and the Levi-Civita connection 1-forms  $\Gamma^{ab}$ , namely  $C^{ab} = \omega^{ab} - \Gamma^{ab}$ . So, the torsion can be written in terms of the contorsion 1-forms as

$$T^a = C^a_b \wedge e^b.$$

Hence, the contorsion 1-forms are found from the equation below

$$C^{ab} = \frac{1}{2} (-e^d i_{X^a} i_{X^b} T_d - i_{X^b} T^a + i_{X^a} T^b), \quad (8)$$

and by using the torsion found above we arrive at [6]

$$C_{ab} = \frac{1}{8\kappa} v_k \epsilon_{[a}{}^{kqp} R_{bd]qp} e^d + \frac{3k}{8\kappa} A^k \epsilon_{kdba} e^d \quad (9)$$

where  $R_{abcd}$  are the components of the curvature tensor.

By reinserting the torsion into the action, one can obtain the effective action, namely this leads to the action that is written as a sum of two terms  $S = S[\Gamma] + S[C]$ . The first one is the torsion free part and the second one includes contorsion-induced interaction terms. So, the effective action is written as

$$\begin{aligned} S_{eff} &= S(\Gamma) + \kappa \int C_{ac} \wedge C^c_b \wedge *e^{ab} \\ &+ \int \theta [R_{ab}(\Gamma) \wedge D_\Gamma C^{ab} + \frac{1}{2} R_{ab}(\Gamma) \wedge C^a_c \wedge C^{cb} \\ &+ D_\Gamma C_{ab} \wedge C^a_c \wedge C^{cb} + \frac{1}{2} C_{ad} \wedge C^{db} \wedge C^a_c \wedge C^{cb} \\ &+ \Gamma_{ac} \wedge C^c_b \wedge (R^{ab}(\Gamma) + D_\Gamma C^{ab} + C^a_d \wedge C^{db})] \\ &+ \frac{ik}{2} \int *e_a \wedge (\bar{\psi} \gamma^a C\psi - C\bar{\psi} \gamma^a \psi). \end{aligned}$$

Here  $D_\Gamma$  is the covariant exterior derivative with respect to the torsion-free connection  $\Gamma$ ,  $R_{ab}(\Gamma)$  are the curvature 2-forms that are constructed from  $\Gamma$  and  $C = \frac{1}{4} C_{ab} \gamma^a \gamma^b$ .

By using the contorsion found above the effective action can be written as [12]

$$\begin{aligned} S_{eff1} &= S(\Gamma) + \int [\frac{k}{16\kappa} (2A^a v^b i_{X_b} P_a - A^a v_a \mathcal{R}) \\ &+ \frac{3k^2}{32\kappa} A^a A_a - \frac{3k^2}{196\kappa^2} \epsilon^{abcd} v_a A_d \partial_c A_b \\ &- \frac{k^3}{256\kappa^3} v_a A^a A^b A_b] * 1 \end{aligned} \quad (10)$$

where  $P_a$  are Ricci 1-forms and  $\mathcal{R}$  is the curvature scalar. Here the terms including contractions with connection 1-forms are neglected by assuming the Riemann normal coordinates. The four-fermion interaction term comes from the first-order formalism without CS modification. So, the CS modification differs mainly by the interaction term of two fermions with CS velocity  $v_a$  and curvature characteristics. The other terms are suppressed by the higher powers of the Newton's constant  $G$ .

## IV. GENERALIZED CS MODIFIED GRAVITY

As we have seen, in CS modified gravity coupled with fermions, the torsion is not zero. However, in the presence of torsion there is one more topological invariant called Nieh-Yan class related with torsion;

$$N = \int (T^a \wedge T_a - R_{ab} \wedge e^{ab}) \quad (11)$$

and this can also be written as a boundary term;

$$T^a \wedge T_a - R_{ab} \wedge e^{ab} = d(e^a \wedge T_a).$$

Hence, one can generalize the CS modified gravity action by including the Nieh-Yan term besides the Pontryagin term. We will see that this will add a small modification to the terms in the effective action that are not suppressed by the higher powers of the Newton's constant;

$$\begin{aligned} S &= S_{EH} + S_{CS} + S_{NY} \\ &= \kappa \int R^{ab} \wedge *e_{ab} \\ &\quad + \frac{1}{2} \int \theta (R^{ab} \wedge R_{ab} + \frac{2}{l^2} (T^a \wedge T_a - R_{ab} \wedge e^{ab})) \end{aligned} \quad (12)$$

where  $l$  is a constant in the units of length. The factor in front of the Nieh-Yan term is added for writing the action in appropriate units. As before, the last term in the action can be written as

$$\begin{aligned} S_{CS} + S_{NY} &= -\frac{1}{2} \int d\theta \wedge (\omega^{ab} \wedge R_{ab} - \frac{1}{3} \omega^{ab} \wedge \omega_a^c \wedge \omega_{cb} \\ &\quad - \frac{2}{l^2} e^a \wedge T_a). \end{aligned} \quad (13)$$

To find the field equations of the generalized CS modified gravity, we take the variations of the action (12). By varying (12) with respect to  $e^a$  and neglecting the boundary terms, one finds that

$$\delta S = \kappa \int R^{ab} \wedge *e_{abc} \wedge \delta e^c - \frac{2}{l^2} \int \delta e_a \wedge d\theta \wedge T^a.$$

Hence the first field equations are found as

$$\kappa R^{bc} \wedge *e_{bc}^a - \frac{2}{l^2} d\theta \wedge T^a = 0.$$

This differs from the without Nieh-Yan case with the inclusion of torsion and the derivative of the coupling field. On the other hand, varying (12) with respect to  $\omega^{ab}$  leads to

$$\begin{aligned} \delta S &= \kappa \int \delta \omega^{ab} \wedge D *e_{ab} + \int \delta \omega^{ab} \wedge (d\theta \wedge R_{ab} \\ &\quad + \frac{1}{l^2} d\theta \wedge e_{ab}). \end{aligned}$$

So, the second field equations are found as below

$$\kappa D *e_{ab} + d\theta \wedge (R_{ab} + \frac{1}{l^2} e_{ab}) = 0. \quad (14)$$

Existence of the last term which did not show up in the previous case is due to the Nieh-Yan term. In the generalized CS modified gravity, variation of the action with respect to the coupling scalar field leads to a constraint again, but this time the constraint requires that the Pontryagin term and Nieh-Yan term must be equal to each other;

$$R^{ab} \wedge R_{ab} = \frac{2}{l^2} (T^a \wedge T_a - R_{ab} \wedge e^{ab}).$$

Now, we can see the effects of the Nieh-Yan term on the CS modified gravity coupled with fermions. In the fermion coupled generalized CS modified gravity, there will be an additional contribution to the torsion. In this case we have the action

$$S = S_{EH} + S_{CS} + S_{NY} + S_D.$$

Effect of the inclusion of matter to the action on the first field equations is only the addition of the stress-energy forms of matter  $\tau^a$ ;

$$\kappa R^{bc} \wedge *e_{bc}^a - \frac{2}{l^2} d\theta \wedge T^a + \tau^a = 0. \quad (15)$$

The second field equations also change because of the existence of matter with spin. In this case equation (14) turns to

$$\kappa D *e_{ab} + d\theta \wedge (R_{ab} + \frac{1}{l^2} e_{ab}) + \frac{k}{4} A^c e_{cab} = 0. \quad (16)$$

Equation (16) leads to a non-zero torsion and it is given in terms of the curvature 2-forms, the CS velocity and the axial current by

$$T_a = \frac{1}{2\kappa} v^b \epsilon_{bapq} (R^{pq} + \frac{1}{l^2} e^{pq}) + \frac{k}{8\kappa} A^b \epsilon_{bapq} e^{pq} + O(v^2) \quad (17)$$

Hence, in the light of (8), the contorsion 1-forms are found as (by neglecting the terms that are second order in  $v$ )

$$\begin{aligned} C_{ab} &= \frac{1}{8\kappa} v_c \epsilon_{[a}^{cqp} R_{bd]qp} e^d + \frac{3}{2\kappa l^2} v^c \epsilon_{cdba} e^d \\ &\quad + \frac{3ik}{8\kappa} A^c \epsilon_{cdba} e^d. \end{aligned} \quad (18)$$

By reinserting the torsion into the action, one can obtain the effective action and see the differences from the CS modified gravity. The extra terms that are originated from the Nieh-Yan term in the effective action can be written as below (by neglecting the terms that includes contractions with connection 1-forms);

$$\begin{aligned} S_{eff2} &= S_{eff1} + \frac{54k}{\kappa l^2} \int A^a v_a * 1 \\ &\quad + \frac{9}{8} \frac{k}{\kappa^2 l^2} \int [A^k v_k R_{abpq} \epsilon^{abpq} \\ &\quad + (A_k v^b + A^b v_k) R_{abpq} \epsilon^{pqka}] * 1 \\ &\quad - \frac{27}{32} \frac{k^2}{\kappa^3 l^2} \int [5\partial_a A^a A^b v_b - \partial_a A_b (A^a v^b + A^b v^a)] * 1 \end{aligned}$$

Hence, only the first extra term is at the order of  $G$  and all the other extra terms are suppressed by at least the square of the Newton's constant. The main difference between the contributions for the effective theories of CS modified and generalized CS modified theories is the term that includes two-fermion and CS velocity interaction without curvature terms;

$$S_{ext} = 54 \frac{k}{\kappa l^2} \int A^a v_a * 1. \quad (19)$$

The other differences occur only on the  $G^2$  order and can be neglected.

## V. CONCLUSION

In the first-order formalism of CS modified gravity, the non-zero torsion is unavoidable. So, the main motivation of CS modified gravity, namely the anomaly cancelation and adding a term to the action which is the Pontryagin invariant coupled with an external scalar field, must be considered in torsional space-time. In this case, the Nieh-Yan term can also be added to the action as a topological invariant which arises in the chiral anomaly with torsion.

It is known that in first-order CS modified gravity, the new effect different from the zero torsion case is the two-fermion interaction with CS velocity contracted by curvature characteristics. This effect can arise in large fermion current regions with space-time having large curvature characteristics such as merging binary neutron stars and neutron star-black hole systems [12].

By considering Nieh-Yan term in the first-order CS

modified gravity, we have arrived at the generalized CS modified gravity. In this case, we have all the correction terms that arise in CS modified gravity and some extra terms resulting from the Nieh-Yan invariant. The main difference, which is first order in  $G$ , is the term of two-fermion interaction with CS velocity without curvature characteristics. This extra term can have effects in large fermion current regions which do not need to be high curvature space-time regions.

## Acknowledgments

I would like to thank Özgür Açıık and Çetin Sentürk for useful discussions during this work. This work was supported in part by the Scientific and Technical Research Council of Turkey (TÜBİTAK).

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